

**$n$ -COMPLETE ALGEBRAS AND MCKAY QUIVERS**

TONGLIANG ZHANG, DEREN LUO, AND LIJING ZHENG\*

**ABSTRACT.** Let  $\Gamma^n$  be the cone of an  $(n-1)$ -complete algebra over an algebraically closed field  $k$ . In this paper, we prove that if the bound quiver  $(Q_n, \rho_n)$  of  $\Gamma^n$  is a truncation from the bound McKay quiver  $(Q_G, \rho_G)$  of a finite subgroup  $G$  of  $GL(n, k)$ , the bound quiver  $(Q_{n+1}, \rho_{n+1})$  of  $\Gamma^{n+1}$ , the cone of  $\Gamma^n$ , is a truncation from the bound McKay quiver  $(Q_{\tilde{G}}, \rho_{\tilde{G}})$  of  $\tilde{G}$ , where  $\tilde{G} \cong G \times \mathbb{Z}_m$  for some  $m \in \mathbb{N}$ .

Recently, Iyama introduced  $n$ -cluster tilting subcategories and developed higher Auslander-Reiten theory ([12]). In [10], he introduced and characterized a class of higher representation algebras,  $n$ -complete algebras. Such algebras are preserved under cone constructions. He also proved that  $n$ -Auslander absolutely  $n$ -complete algebra are constructed by iterative cone construction starting from some path algebra of quiver of type  $A_r$  with linear orientation. Guo proved that such algebras can be obtained from a truncation of the McKay quivers of some abelian groups ([5]).

In this paper, we generalize the results of Guo and prove the following result: Let  $\Gamma^n$  be the cone of an  $(n-1)$ -complete algebra, if the bound quiver  $(Q_n, \rho_n)$  of  $\Gamma^n$  is a truncation from the bound McKay quiver  $(Q_G, \rho_G)$  of a finite subgroup  $G$  of  $GL(n, k)$ , then there exists a positive integer  $m$  such that the bound quiver  $(Q_{n+1}, \rho_{n+1})$  of  $\Gamma^{n+1}$  is a truncation from the bound McKay quiver  $(Q_{\tilde{G}}, \rho_{\tilde{G}})$  of a finite subgroup  $\tilde{G} \cong G \times \mathbb{Z}_m$  in  $GL(n+1, k)$ .

The paper is organized as follows. In Section 1, we shall recall some basic definitions and facts needed for  $n$ -complete algebras, McKay quivers and trivial extensions of graded self-injective algebras. Then we describe the bound McKay quivers using twisted trivial extensions in Section 2 and Section 3. Our main theorem will be stated and proved in Section 4.

**1. PRELIMINARIES**

Throughout this paper,  $k$  is an algebraically closed field of characteristic 0. Let  $\Lambda$  be an algebra over  $k$ . Denote by  $\text{mod } \Lambda$  the category of finitely generated left  $\Lambda$ -modules, and for  $M \in \text{mod } \Lambda$ , denote by  $\text{add } M$  the full subcategory of  $\text{mod } \Lambda$  consisting of direct summands of finite direct sums of copies of  $M$ , and denote by  $D$  the standard  $k$ -duality  $\text{Hom}_k(-, k)$ .

---

2010 *Mathematics Subject Classification.* Primary 16G10; Secondary 16S34, 16P90.

*Key words and phrases.*  $n$ -complete algebra; McKay quiver; returning arrows; covering spaces.

\* E-mail: zhengliling817@163.com

This work is partly supported by Natural Science Foundation of China #11271119 and by Hunan Provincial Innovation Foundation For Postgraduate(CX2013B216 and CX2014B189).

**1.1.  $n$ -complete algebra.** In [12], Iyama introduced and studied  $n$ -complete algebra. Let  $\Lambda$  be a finite dimensional  $k$ -algebra. Let  $\tau_n = D\text{Tr}\Omega^{n-1}$  be the  $n$ -Auslander-Reiten translation, the subcategory  $\mathcal{M} = \mathcal{M}_n(D\Lambda) = \text{add}\{\tau_n^i(D\Lambda) | i \geq 0\}$  of  $\text{mod } \Lambda$  is called the  $\tau_n$ -closure of  $D\Lambda$ . Let

$$\begin{aligned}\mathcal{I}(\mathcal{M}) &= \text{add } D\Lambda, \\ \mathcal{P}(\mathcal{M}) &= \{X \in \mathcal{M} \mid \tau_n X = 0\}, \\ \mathcal{M}_P &= \{X \in \mathcal{M} \mid X \text{ has no non-zero summands in } \mathcal{P}(\mathcal{M})\}, \\ \mathcal{M}_I &= \{X \in \mathcal{M} \mid X \text{ has no non-zero summands in } \mathcal{I}(\mathcal{M})\}.\end{aligned}$$

$\Lambda$  is called  $n$ -complete if the global dimension of  $\Lambda$ ,  $\text{gl.dim } \Lambda \leq n$ , and the following conditions (1) – (3) are satisfied.

- (1). There exists a tilting  $\Lambda$ -module  $T$  satisfying  $\mathcal{P}(\mathcal{M}) = \text{add } T$ ,
- (2).  $\mathcal{M}$  is an  $n$ -cluster titing subcategory of  $T^\perp = \{X \in \mathcal{M} \mid \text{Ext}_\Lambda^i(T, X) = 0 (0 < i) \}$ ,
- (3).  $\text{Ext}_\Lambda^i(\mathcal{M}_P, \Lambda) = 0$  for any  $0 < i < n$ .

In this case,  $\Gamma := \text{End}_\Lambda(\bigoplus_{t \geq 0} \tau_n^t(D\Lambda))$  is called the *cone* of  $\Lambda$ .

Let  $\Lambda$  be an  $n$ -complete algebra and  $\mathcal{M} = \mathcal{M}_n(D\Lambda)$  the  $\tau_n$ -closure of  $D\Lambda$ , we denote by  $J_{\mathcal{M}}$  the Jacobson radical of  $\mathcal{M}([1])$ . Define quiver  $(Q_0 = Q_0(\mathcal{M}), Q_1 = Q_1(\mathcal{M}))$  and a map  $\tau_n$  as follows:

- (1).  $Q_0$  (respectively,  $Q_P$  and  $Q_I$ ) is the set of indecomposable objects in  $\mathcal{M}$  (respectively,  $\mathcal{M}_P$  and  $\mathcal{M}_I$ ).
- (2). For  $X, Y \in Q_0$ , put  $d_{XY} = \dim(J_{\mathcal{M}}(X, Y)/J_{\mathcal{M}}^2(X, Y))$  and draw  $d_{XY}$  arrows from  $X$  to  $Y$ .
- (3). A map  $\tau_n : Q_P \rightarrow Q_I$  is given by  $\tau_n : \mathcal{M}_P \rightarrow \mathcal{M}_I$ .

$(Q_0, Q_1, \tau_n)$  is a weak translation quiver in the sense of [12] and is called the *Auslander-Reiten quiver* of  $\mathcal{M}$ .

Let  $\Gamma^n$  be the cone of an  $(n-1)$ -complete algebra  $\Gamma^{n-1}$ , and let  $(Q_{n,0}, Q_{n,1}, \tau_n)$  be the Auslander-Reiten quiver of  $\mathcal{M}_{n-1}(D\Gamma^{n-1})$ . By Lemma 6.4 of [12],  $\Gamma^n$  is given by the bound quiver  $(Q_n, \rho_n)$ , that is  $\Gamma^n \cong kQ/(\rho_n)$  for some relation  $\rho_n$ . We say that  $\Gamma^n$  is given by the *bound quiver*  $(Q_n, \rho_n)$  with  $n$ -Auslander-Reiten translation  $\tau_n$ . The bound Auslander-Reiten quiver of  $\mathcal{M}_n(D\Gamma^n)$  is  $(Q_{n+1}, \rho_{n+1}, \tau_{n+1})$ .

Iyama proved the following theorem which described the relationship between the bound quivers of  $n$ -complete algebra  $\Gamma^n$  and its cone  $\Gamma^{n+1}$  with Auslander-Reiten translations ([12], Theorem 6.7).

**Theorem 1.1.** *Let  $\Gamma^n$  be an  $n$ -complete algebra which is the cone of an  $(n-1)$ -complete algebra  $\Gamma^{n-1}$  and let  $(Q_n, \rho_n)$  be the bound quiver of  $\Gamma^{n-1}$  with  $n$ -Auslander-Reiten translations  $\tau_n$ . Then the bound Auslander-Reiten quiver  $(Q_{n+1}, \rho_{n+1})$  with  $n$ -Auslander-Reiten translation  $\tau_{n+1}$  of  $\Gamma^n$  is given by the followings:*

- The vertex set

$$Q_{n+1,0} = \{(x, d) \mid x \in Q_{n,0}, d \geq 0, \tau_n^d x \neq 0\},$$

with

$$Q_{n+1,P} = \{(x, d) \mid x \in Q_{n,0}, \tau_n^{d+1} x = 0\}$$

$$Q_{n+1,I} = \{(x, 0) \mid x \in Q_{n,0}\}.$$

- The arrow set  $Q_{n+1,1}$  consists of two types of arrows:

- The set of arrows of the first type obtained from arrows in  $Q_{n,1}$ :  
 $\{(\alpha, d) : (x, d) \rightarrow (y, d) | \alpha : x \rightarrow y \in Q_{n,1}, \text{ and } (x, d), (y, d) \in Q_{n+1,0}\}.$
- The set of arrows of the second type obtained by the  $n$ -Auslander-Reiten translation  $\tau_n$ :  
 $\{(x, d)_1 : (x, d) \rightarrow (\tau_n x, d - 1) | (x, d) \in Q_{n+1,0}, d > 0\}.$

- The relation set

$$\rho_{n+1} = \{(r, d) | r \in \rho_n\} \cup \{(\alpha, d - 1)(x, d)_1 - (\tau_n^- y, d)_1(\tau_n^- \alpha, d) \mid \alpha : \tau_n x \rightarrow y \in Q_{n,1}, d > 0\},$$

here  $(r, d)$  is defined as follow: if  $p = \alpha_l \cdots \alpha_1$  is a path in  $Q_G$ , write  $(p, t) = (\alpha_l, t) \cdots (\alpha_1, d)$  and if  $r = \sum_v h_v p_v$  for paths  $p_v$  of  $Q_G$  and  $h_v \in k$ , write  $(r, d) = \sum_v h_v (p_v, d)$ .

- The  $n + 1$ -Auslander-Reiten translation:

$$\tau_{n+1} : Q_{n+1,P} \rightarrow Q_{n+1,I}, \quad \tau_{n+1}(x, d) = (x, d + 1).$$

**1.2. McKay quiver.** McKay quiver was introduced in 1980 by John McKay for a finite subgroup of the general linear group. Let  $s$  be a positive integer and let  $G \subset GL(s, k) = GL(V)$  be a finite subgroup, here  $V$  is an  $s$ -dimensional vector space over  $k$ . Let  $\{S_i | i = 1, 2, \dots, n\}$  be a complete set of irreducible representations of  $G$  over  $k$ . For each  $S_i$ , decompose the tensor product  $V \otimes S_i$  as a direct sum of irreducible representations, write

$$V \otimes S_i = \bigoplus_j a_{i,j} S_j, \quad i = 1, \dots, n,$$

here  $a_{i,j} S_j$  is a direct sum of  $a_{i,j}$  copies of  $S_j$ . The McKay quiver  $Q = Q(G)$  of  $G$  is defined the quiver with the vertex set  $Q_0(G)$  the set of (the indices of) the isomorphism classes of irreducible representations of  $G$ , and there are  $a_{i,j}$  arrows from the vertex  $i$  to the vertex  $j$  in the arrow set  $Q_1(G)$ .

In [6], Guo and Martínez-Villa proved the McKay quiver of  $G$  is the quiver of a pair of skew group algebras  $k[V] * G$  and  $\wedge V * G$ , where  $k[V]$  and  $\wedge V$  are respectively the symmetric algebra and exterior algebra of the vector space  $V$ . Thus there is idempotent  $e$  and  $\bar{e}$  in  $k[V] * G$  and  $\wedge V * G$  respectively, such that their basic algebras  $V(G) = ek[V] * Ge \simeq kQ(G)/(\rho(G))$  and  $\Lambda(G) = \bar{e} \wedge V * G \bar{e} \simeq kQ(G)/(\theta(G))$  for the path algebra  $kQ(G)$  of  $Q(G)$  and for some relation sets  $\rho(G)$  and  $\theta(G)$ , respectively. Since  $\wedge V * G$  is a self-injective algebra, we have a *Nakayama translation*  $\nu$  on the McKay quiver  $Q(G)$ , induced by the Nakayama functor. We call the bound quiver  $(Q_0(G), Q_1(G), \rho(G))$  the *bound quiver* of  $G$ .

It follows from Theorem 2.1 of [6] that the basic algebras  $V(G) = V(G)_0 + V(G)_1 + \cdots$  of  $k[V] * G$  and the basic algebra  $\Lambda(G) = \Lambda(G)_0 + \Lambda(G)_1 + \cdots + \Lambda_s$  of  $\wedge V * G$  are quadratic dual. They share the same quiver  $Q_G$ , and we have  $V(G)_1 = \Lambda(G)_1 = kQ_{G,1}$  is the space spanned by the arrows in  $Q_G$ . The arrows form a basis of  $V(G)_1$  and dual basis of  $\Lambda(G)_1 = \text{Hom}_k(V(G)_1, k)^{op}$ . That is, we have  $\alpha(\alpha') = \begin{cases} 1 & \alpha = \alpha' \\ 0 & \alpha \neq \alpha' \end{cases}$ , when they are regarded as arrows taken from the quiver of  $\Lambda(G)$  and from the quiver  $V(G)$ , respectively. We have  $\alpha\beta(\alpha'\beta') = \alpha(\alpha')\beta(\beta')$  in the space  $kQ_2$  of paths of length two. Then relation set  $\theta(G)$  and  $\rho(G)$  span orthogonal subspaces in  $kQ_2$ .

In [9], Guo found that a returning arrow appears from  $i$  to  $\nu(i)$  at each vertex  $i$  in the McKay quiver of  $G$  when embedding  $GL(V)$  to  $SL(V')$  for some  $s + 1$ -dimensional space  $V' \supset V$  in some standard manner; and when embedding  $G$  into a group  $\tilde{G}$  in  $GL(V')$  such that  $G = \tilde{G} \cap SL(V')$  and  $\tilde{G}/G$  is a cyclic group of order  $m$ , the McKay quiver of  $\tilde{G}$  is  $m$  copies of the McKay quiver of  $G$  connected by the arrows induced by the returning arrows. Our main theorem is motivated from this construction of McKay quivers. We will use trivial extension of self-injective algebras and quadratic duality to give a concrete description of the bound McKay quiver of  $\tilde{G}$ .

**1.3. Returning arrows for the trivial extensions.** Recall that the *trivial extension*  $\Lambda \ltimes M$  of an algebra  $\Lambda$  by a  $\Lambda$ - $\Lambda$ -bimodule  $M$  is the algebra defined on the vector space  $\Lambda \oplus M$  with the multiplication defined by

$$(a, x)(b, y) = (ab, ay + xb)$$

for  $a, b \in \Lambda$  and  $x, y \in M$ .

Let  $M$  be a right  $\Lambda$ -module and let  $\sigma \in \text{Aut}(\Lambda)$ . By  $M^\sigma$  we denote the right  $\Lambda$ -module such that  $M^\sigma = M$  as  $k$ -vector space and the right operation of an element  $a$  of  $\Lambda$  on  $M^\sigma$  is defined by  $ma = m\sigma(a)$ , for  $m \in M$ . Similarly,  ${}^\sigma N$  is defined for a left  $\Lambda$ -module  $N$  and an element  $\sigma \in \text{Aut}(\Lambda)$ .

Let  $\sigma \in \text{Aut}(\Lambda)$ . Denote by  $D(\Lambda^\sigma)$  the dual  $\Lambda$ -bimodule of  $\Lambda^\sigma$ , and the corresponding trivial extension  $T(\Lambda^\sigma) = \Lambda \ltimes D(\Lambda^\sigma)$  is called the twisted trivial extension algebra of  $\Lambda$ .

Fix an integer  $l \geq 1$ , recall that the bound quiver  $Q$  of a graded self-injective algebra  $\Lambda$  of Loewy length  $l + 1$  is a *stable translation quiver of Loewy length  $l + 1$*  satisfying the following conditions[4]:

- (1). A permutation  $\nu$  is defined on the vertex set of  $Q$ ;
- (2). The maximal bound paths of  $Q$  have the same length  $l$ ;
- (3). For each vertex  $i$ , there is a maximal bound path from  $\nu(i)$  to  $i$ , and there is no bound path of length  $l$  from  $\nu(i)$  to  $j$  for any  $j \neq i$ ;
- (4). Any two maximal bound paths starting at the same vertex are linearly dependent.

$\nu$  is called the Nakayama translation of the stable translation quiver  $Q$ , it is induced by the Nakayama functor.

Let  $\Lambda$  be a finite dimensional graded self-injective algebra given by a stable translation quiver  $(Q, \rho)$  of Loewy length  $l + 1$  with the Nakayama translation  $\nu$ . By [8], there is a map  $\nu$  from  $Q_1$  to  $\bigcup_{i,j} e_j \Lambda_1 e_i$  such that  $\nu(Q_1)$  is a basis of  $\Lambda_1$  which can be extended to a graded automorphism on  $\Lambda$ . Actually,  $\nu$  is the Nakayama automorphism of  $\Lambda$  induced by Nakayama functor on  $\text{mod } \Lambda$ . The following quiver  $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1)$  is called the returning arrow quiver of  $(Q, \rho)$ [8], where the vertex set  $\tilde{Q}_0 = Q_0$ , the arrow set  $\tilde{Q}_1 = Q_1 \cup \{\alpha_i : i \rightarrow \nu(i) \mid i \in Q_0\}$ ;

Recall that an automorphism  $\sigma$  is called *nice* if it preserves idempotents, that is,  $\sigma(e) = e$  for all the idempotents of  $\Lambda$ . The following theorem is proved in [8].

**Theorem 1.2.** *Let  $\Lambda$  be a finite dimensional graded self-injective algebra given by a stable translation quiver  $(Q, \rho)$  of Loewy length  $l + 1$  with the Nakayama translation  $\nu$ . If  $\sigma$  is a nice graded automorphism of  $\Lambda$ , then the twisted trivial extension  $T(\Lambda^\sigma)$  is given by the bound quiver  $(\tilde{Q}, \tilde{\rho}^\sigma)$ , where  $\tilde{\rho}^\sigma = \rho \cup \{\alpha_{\nu i} \alpha_i \mid i \in Q_0\} \cup \{\alpha_j \beta - \sigma^{-1} \nu(\beta) \alpha_i \mid \beta : i \rightarrow j \in Q_1\}$ .*

Let  $G$  be a finite subgroup of  $\text{Aut}_k(\Lambda)$ , and let  $\Lambda * G$  be the skew group algebra of  $\Lambda$ , assume that  $\sigma \in \text{Aut}_k(\Lambda)$  satisfying  $g\sigma = \sigma g$  for  $g \in G$ , then  $\sigma$  induces a automorphism  $\tilde{\sigma}$  of  $\Lambda * G$  such that  $\tilde{\sigma}(a * g) = \sigma(a) * g$  for  $a \in \Lambda, g \in G$ . We define  $g(a, \varphi) = (g(a), \varphi g^{-1})$  for  $(a, \varphi) \in T(\Lambda^\sigma)$ . Since  $g\sigma = \sigma g$ ,  $g$  can be lifted a automorphism of  $T(\Lambda^\sigma)$ . So  $G$  is a finite subgroup of  $\text{Aut}_k(T(\Lambda^\sigma))$  and  $T(\Lambda^\sigma) * G$  makes sense.

**Lemma 1.3.** ([16, Lemma 2.2]) *Let  $\Lambda$  be a finite dimensional algebra,  $G$  a finite subgroup of  $\text{Aut}_k(\Lambda)$ , assume that  $\sigma \in \text{Aut}_k(\Lambda)$  satisfying  $g\sigma = \sigma g$  for each  $g \in G$ . Then we have a  $k$ -algebra isomorphism.*

$$T(\Lambda^\sigma) * G \cong T((\Lambda * G)^{\tilde{\sigma}}).$$

**Remark 1.4.** *By the proof of Theorem 2.3 in [8], we know that the twisted trivial extension  $T(\Lambda^\sigma)$  can also be given by  $(\tilde{Q}, \hat{\theta}^\sigma)$ , where*

$$\hat{\theta}^\sigma = \rho \cup \{\alpha_{\tau i} \alpha_i \mid i \in Q_0\} \cup \{\beta \alpha_{\tau^{-1}(i)} - \alpha_{\tau^{-1}(j)} \nu^{-1} \sigma(\beta) \mid \beta : i \rightarrow j \in Q_1\}.$$

## 2. THE BOUND MCKAY QUIVER WITH RETURNING ARROWS

Let  $V$  be an  $s$ -dimensional vector space over  $k$ , and let  $V'$  be an  $s+1$ -dimensional vector space containing  $V$ . Let  $G \subset GL(V)$  be a finite subgroup.  $GL(V) \subset SL(V')$  via the natural embedding  $g \rightarrow \begin{pmatrix} g & \\ & \det^{-1} g \end{pmatrix}$ , where  $\det$  is the determinant map. Let  $G'$  be the image of  $G$ . The McKay quiver  $Q(G')$  is obtained from  $Q(G)$  by inserting an arrow  $\beta : \nu i \rightarrow i$  for each vertex  $i$  in  $Q(G)$ .

Let  $\Lambda(G)$  and  $\Lambda'(G)$  be the basic versions of the skew group algebras  $\Lambda V * G$  and  $\Lambda V' * G'$ . We first study the bound quiver of  $\Lambda'(G)$  using trivial extension of graded self-injective algebras.

We need a lemma comparing certain twisted trivial extensions of a graded algebra with its basic algebra. Let  $A = A_0 + A_1 + \cdots + A_l$  be a finite dimensional graded algebra such that  $A_0$  is semi-simple and  $A$  is generated by  $A_0$  and  $A_1$ . Let  $B = B_0 + B_1 + \cdots + B_l$  be a basic version of  $A$ , that is,  $B$  is a basic algebra Morita equivalent to  $A$ . Then  $B_0$  is a direct sum of finite copies of  $k$ , and generated by  $B_0$  and  $B_1$ .

Let  $\sigma_0$  be the graded automorphism of  $A$  induced by the map

$$\sigma_0 : \gamma \mapsto (-1)^n \gamma$$

on  $A_1$ . Then  $\sigma_0$  induces an automorphism of  $B$ , which we also denote by  $\sigma_0$ .

**Lemma 2.1.**  *$T(A^{\sigma_0})$  and  $T(B^{\sigma_0})$  are Morita equivalent.*

*Proof.* Take  $e \in A_0$  to be an idempotent such that  $B = eAe$  is the basic algebra of  $A$ , it follows from Corollary 6.10 of [1] that  $F = \text{Hom}(Ae, -) : \text{mod} A \rightarrow \text{mod} B$  is an equivalence of categories. Then by Theorem 6.8 of [15], there is a projective generator  ${}_B P = F(A) = eA$  such that  $\text{End}_B(P) \cong A$  as  $k$ -algebras. Clearly  $P$  is a graded  $B$ - $A$ -bimodule. By [2] there are integers  $m, n$  and modules  $B', P'$  with the following left  $B$ -module isomorphisms

$$P^n \cong B \oplus B', B^m \cong P \oplus P'.$$

Thus we have the left  $T(B^{\sigma_0})$ -module isomorphisms

$$(T(B^{\sigma_0}) \otimes_B P)^n \cong T(B^{\sigma_0}) \oplus (T(B^{\sigma_0}) \otimes_B B')$$

and

$$(T(B^{\sigma_0}))^n \cong (T(B^{\sigma_0}) \otimes_B P) \oplus ((T(B^{\sigma_0}) \otimes_B P')).$$

Thus  $T(B^{\sigma_0}) \otimes_B P$  is a projective generator of  $\text{mod } T(B^{\sigma_0})$ , write it as  $\tilde{P}$ .  
By Proposition 1.8 and the claim above Proposition 1.13 of [3], we have

$$\text{End}_{T(B^{\sigma_0})}(\tilde{P}) \cong \text{End}_B(P) \ltimes \text{Hom}_B(P, D(B^{\sigma_0}) \otimes P).$$

It follows from Lemma 5 of [11] that

$${}_A D(A)_A \cong {}_A \text{Hom}_B({}_B P_A, {}_B D(B) \otimes_B P_A)_A.$$

Observe that

$${}_A D(A^{\sigma_0})_A \cong {}_A \text{Hom}_B({}_B P_A^{\sigma_0}, {}_B D(B) \otimes_B P_A)_A.$$

Define

$$\pi : {}_A \text{Hom}_B({}_B P_A^{\sigma_0}, {}_B D(B)_A \otimes_B P_A)_A \rightarrow {}_A \text{Hom}_B({}_B P_A, {}_B D(B^{\sigma_0}) \otimes_B P_A)_A$$

by  $\pi(f)(p) = \sum_i (-1)^i f(p_i)$ , for  $f \in {}_A \text{Hom}_B({}_B P_A^{\sigma_0}, {}_B D(B)_A \otimes_B P_A)_A, p \in P$ ,  
 $p = \sum_i p_i$  with  $p_i \in P_i$ .

Then  $\pi$  is an  $A$ - $A$ -bimodule isomorphism.

Thus we have an algebra isomorphism  $T(A^{\sigma_0}) \cong \text{End}_B(P) \ltimes \text{Hom}_B(P, D(B^{\sigma_0}) \otimes P)$ . By Theorem 6.7 of [15],  $T(A^{\sigma_0})$  and  $T(B^{\sigma_0})$  are Morita equivalent.  $\square$

Let  $\varepsilon$  be a graded automorphism of  $\wedge V$  induced by the linear map

$$\varepsilon : x \mapsto -x$$

on the vector space  $V$ .

By Proposition 3.1 of [15], the Nakayama automorphism  $\nu$  of  $\wedge V$  is induced by the linear map defined by  $\nu(x) = (-1)^{n-1}x$  for  $x \in V$ . Let  $\sigma = \varepsilon\nu$ , then  $\sigma$  is a graded automorphism of  $\wedge V$ . For  $g = (r_{ij})_{n \times n} \in G$ , we have

$$\sigma g(x_i) = g\sigma(x_i) = (-1)^n \sum_{j=1}^n r_{ij} x_j,$$

for  $1 \leq i \leq n$ . Therefore  $\sigma g = g\sigma$  on  $\wedge V$ . By the argument above Lemma 1.3,  $G$  can be regarded as an automorphism group of  $T(\wedge V^\sigma)$  and  $\sigma$  induces an automorphism  $\tilde{\sigma}$  on  $\wedge V * G$  such that

$$\tilde{\sigma}(x_i * g) = \sigma(x_i) * g = (-1)^n (x_i * g), \quad \tilde{\sigma}(p * g) = \sigma(p) * g = p * g$$

for  $p \in (\wedge V)_0 = k, 1 \leq i \leq n$ , and hence  $\tilde{\sigma}$  is a nice automorphism of  $\wedge V * G$ .

We have the following proposition.

**Proposition 2.2.**  $\wedge V' * G' \cong T((\wedge V * G)^{\tilde{\sigma}})$  as algebras.

*Proof.* Note that  $D(\wedge V)$  is generated by  $(\wedge V)_n^* = D((\wedge V)_n)$  as  $\wedge V$ - $\wedge V$ -bimodule, and assume that  $0 \neq \varphi \in (\wedge V)_n^*$ . As an algebra,  $T(\wedge V^\sigma)$  is generated by  $(\wedge V)_0$  and  $(\wedge V)_1 + (\wedge V)_n^*$ , with the additional relation  $\varphi^2 = 0$  and  $\varphi x + x\varphi = 0$ , by Proposition 2.5 of [8]. This shows  $T(\wedge V^\sigma) \simeq \wedge V'$  and  $T(\wedge V^\sigma) * G \simeq \wedge V' * G'$ . So we get  $\wedge V' * G' \cong T((\wedge V * G)^{\tilde{\sigma}})$  by Lemma 1.3.  $\square$

The following proposition recovers and generalizes Theorem 3.1 of [9].

**Proposition 2.3.** *Suppose that  $(Q_G, \theta_G)$  is the bound quiver of  $\Lambda(G)$ , and let  $\nu$  be the Nakayama automorphism on  $\Lambda(G)$ . The bound quiver  $(Q_{G'}, \theta_{G'})$  of  $\wedge V' * G'$  is given by the followings:*

1.  $Q_{G'}$  is obtained from  $Q_G$  by adding an arrow  $\beta_i : i \rightarrow \nu i$  for each vertex  $i \in Q_{G,0}$ .
2.  $\theta_{G'} = \theta_G \cup \{\beta_{\nu i} \beta_i \mid i \in Q_{G,0}\} \cup \{\alpha \beta_{\nu^{-1}i} - \beta_{\nu^{-1}j} \nu^{-1} \tilde{\sigma}(\alpha) \mid \alpha : i \rightarrow j \in Q_{G,1}\}$ .

*Proof.* By Proposition 2.2,  $\wedge V' * G' \cong T((\wedge V * G)^{\tilde{\sigma}})$ . By Lemma 2.1, we have that  $T(\Lambda(G)^{\tilde{\sigma}})$  is the basic algebra of  $T((\wedge V * G)^{\tilde{\sigma}})$ , and  $T(\Lambda(G)^{\tilde{\sigma}}) \cong kQ_{G'}/(\theta_{G'})$ .  $\Lambda(G)$  is a finite dimensional graded self-injective algebra with stable translation quiver  $(Q_G, \theta)$  with the Nakayama translation induces by  $\nu$ . So by Theorem 1.2, we get that  $Q_{G'}$  is obtained from  $Q_G$  by adding an arrow  $\beta_i : i \rightarrow \nu(j)$  for each vertex  $i \in Q_{G,0}$ , and by Theorem 1.2 and Remark 1.4,

$$\theta_{G'} = \theta_G \cup \{\beta_{\nu i} \beta_i \mid i \in Q_{G,0}\} \cup \{\alpha \beta_{\nu^{-1}i} - \beta_{\nu^{-1}j} \nu^{-1} \tilde{\sigma}(\alpha) \mid \alpha : i \rightarrow j \in Q_{G,1}\}.$$

□

Furthermore, we have the following proposition on the bound McKay quiver.

**Proposition 2.4.** *Let  $(Q_G, \theta_G)$  be the bound quiver of  $\wedge V * G$ , then the relation set of the bound McKay quiver  $(Q_{G'}, \rho_{G'})$  of  $G'$  is given by*

$$\rho_{G'} = \rho_G \cup \{\alpha \beta_{\nu^{-1}(i)} - \beta_{\nu^{-1}(j)} \nu^{-1}(\alpha) \mid \alpha : i \rightarrow j \in Q_{G,1}\}.$$

*Proof.* Let  $Q_{G,2}$  and  $Q_{G',2}$  be the sets of the paths of length 2 of  $Q_G$  and of  $Q_{G'}$ , respectively. Since  $Q_{G,1}$  is a subset of  $Q_{G',1}$ , we have  $\rho(G) \subset \rho(G')$ ,  $\theta(G) \subset \theta(G')$  and  $kQ_{G,2} \subset kQ_{G',2}$ . Since  $V(G)$  and  $\Lambda(G)$  are quadratic dual, and  $V(G)$  and  $\Lambda(G)$  are quadratic dual,  $\rho(G)$  and  $\theta(G)$  span orthogonal subspaces in  $kQ_{G,2}$ , and  $\rho(G')$  and  $\theta(G')$  span orthogonal subspaces in  $kQ_{G',2}$ . For each pair  $i, j$  of vertices, the subsets  $e_j \theta_{ij}(G) e_i$  and  $e_j \rho(G) e_i$  span of orthogonal subspaces in  $e_j kQ_{G,2} e_i$ , and the subsets  $e_j \theta_{ij}(G') e_i$  and  $e_j \rho(G') e_i$  span of orthogonal subspaces in  $e_j kQ_{G',2} e_i$ .

Fix a vertex  $i$ , the set the paths of length 2 from  $i$  to  $j$  of  $Q_{G'}$  is

$$e_j Q_{G',2} e_i = e_j Q_{G,2} e_i \cup \{\beta_{\nu i} \beta_i\} \cup \{\beta_{\nu^{-1}j} \alpha, \nu(\alpha)' \beta_i \mid \alpha \in e_{\nu^{-1}j} Q_1 e_i\}$$

if  $j = \nu^2(i)$ , and is

$$e_j Q_{G',2} e_i = e_j Q_{G,2} e_i \cup \{\beta_{\nu(i)} \beta_i\} \cup \{\beta_{\nu^{-1}(j)} \alpha, \nu(\alpha)' \beta_i \mid \alpha \in e_{\nu^{-1}(j)} Q_1 e_i\}$$

if  $j \neq \nu^2(i)$ . Thus  $e_j \rho(G') e_i$  span an orthogonal subspace of  $e_j \theta(G') e_i$  since

$$\begin{aligned} & (\alpha \beta_{\nu^{-1}(i)} - \beta_{\nu^{-1}(j)} \nu^{-1} \tilde{\sigma}(\alpha)) (\alpha \beta_{\nu^{-1}(i)} + \beta_{\nu^{-1}(j)} \nu^{-1} \tilde{\sigma}(\alpha)) \\ &= \alpha(\alpha) \beta_{\nu^{-1}(i)} (\beta_{\nu^{-1}(i)} - \beta_{\nu^{-1}(j)} \nu^{-1} \tilde{\sigma}(\alpha)) (\beta_{\nu^{-1}(j)} \nu^{-1} \tilde{\sigma}(\alpha)) \\ &= 0 \end{aligned}$$

Since  $\tilde{\sigma}(\alpha) = -\alpha$  for all  $\alpha \in Q_{G,1}$ , this shows that

$$\rho_{G'} = \rho_G \cup \{\alpha \beta_{\nu^{-1}(i)} - \beta_{\nu^{-1}j} \nu^{-1} \alpha \mid \alpha : i \rightarrow j \in Q_{G,1}\}$$

is the relation set for  $V(G')$ .

□

## 3. BOUND MCKAY QUIVERS FOR CYCLIC EXTENSION

By embedding  $GL(V)$  into  $SL(V')$ , we obtain returning arrows in the McKay quiver of a group. We need smash product construction for the algebra to change the returning arrows into connecting arrows between the copies of the original McKay quiver. These connecting arrows are the arrows of type  $(x, d)_1 : (x, d) \rightarrow (\tau_n x, d-1)$  in the Auslander-Reiten quiver of the cone (Theorem 1.1). Such construction is realized by a direct product of group  $G'$  with a cyclic group. Let  $H$  be a finite group, and let  $\Lambda$  be an  $H$ -graded algebra. Recall that the smash product  $\Lambda \# kH^*$  is the free  $\Lambda$ -module with basis  $H^* = \{\delta_g \mid g \in H\}$ , and the multiplication is defined by

$$a\delta_g \cdot b\delta_h = ab_{gh^{-1}}\delta_h,$$

for  $a, b \in \Lambda$ , where  $b_{gh^{-1}}$  is the homogeneous component of degree  $gh^{-1}$  of  $b$  in  $\Lambda$ .

Let  $\xi_m \in k$  be a primitive  $m$ -th root of unity and let

$$\omega_m = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \xi_m \end{pmatrix}_{(s+1) \times (s+1)}$$

Let  $C_m = \langle \xi_m \rangle$  be the cyclic subgroup of  $GL(V')$  generated by  $\omega_m$  and let  $\tilde{G} = GC_m = G \times C_m$ , then  $\tilde{G} \cap SL(V') = G$ . Let  $\widehat{C_m} = \text{Hom}(C_m, k^*) \cong \mathbb{Z}_m$  be the dual group of  $C_m$ . For  $\lambda_i \in \text{Hom}(C_m, k^*)$  such that  $\lambda_i(\omega_m) = \xi_m^i$ , then  $\lambda_i = \lambda_1^i$  and  $\widehat{C_m}$  is the cyclic group generated by  $\lambda_1$ .

Let  $V' = V + V_1$  is a direct sum of subspaces, and  $V_1$  is an one dimensional subspace of spanned by  $y$ . The action of  $C_m$  on  $V'$  induce a  $C_m$ -grading on  $\wedge V' * G'$  with such that  $a \in (\wedge V' * G')_{\omega^i}$  if and only if  $\omega(a) = \xi^i a$ . Thus  $\wedge V' * G' = (\wedge V' * G')_{\omega_0} + (\wedge V' * G')_{\omega_1}$ , with  $(\wedge V' * G')_{\omega_0} = \wedge V * G'$  and  $(\wedge V' * G')_{\omega_1} = y(\wedge V * G')$ , is a  $C_m$ -graded algebra. Now it follows from [14] that

$$\wedge V' * \tilde{G} \cong \wedge V' * (G' \times C_m) \cong (\wedge V' * G') * C_m,$$

and by [13],  $(\wedge V' * G') * C_m \cong (\wedge V' * G') \# k\widehat{C_m}^*$ .

Clearly, if  $e_i$  is a primitive idempotent of  $\wedge V' * G'$ , then for each  $j \in \mathbb{Z}/m\mathbb{Z}$ ,  $e_i \delta_j$  is a primitive idempotent of  $(\wedge V' * G') \# k\widehat{C_m}^*$ , and we have the following lemma.

**Lemma 3.1.** *Suppose that  $e_i$  and  $e'_i$  are primitive idempotents of  $\wedge V' * G'$ , and  $e_i(\wedge V' * G') \cong e'_i(\wedge V' * G')$  as right  $\wedge V' * G'$ -modules, then for  $j \in \mathbb{Z}/m\mathbb{Z}$ ,  $e_i \delta_j((\wedge V' * G') \# k\widehat{C_m}^*) \cong e'_i \delta_j((\wedge V' * G') \# k\widehat{C_m}^*)$  as right  $\wedge V' * G' \# k\widehat{C_m}^*$ -modules.*

*Proof.* If  $\varphi : e_i(\wedge V' * G') \rightarrow e'_i(\wedge V' * G')$  is an isomorphism as right  $\wedge V' * G'$ -modules with  $\varphi(e_i) = e'_i b$  for  $b \in \wedge V' * G'$ , then

$$\varphi(e_i)_0 \delta_t = e'_i b_0 \delta_t = e'_i \delta_t b \delta_t \in e'_i \delta_t((\wedge V' * G') \# k\widehat{C_m}^*).$$

Set  $\varphi'(e_i \delta_t) = \varphi(e_i)_0 \delta_t$ , we get right  $\wedge V' * G' \# k\widehat{C_m}^*$ -module isomorphism

$$\varphi' : e_i \delta_t((\wedge V' * G') \# k\widehat{C_m}^*) \rightarrow e'_i \delta_t((\wedge V' * G') \# k\widehat{C_m}^*).$$

□



Let  $e = e_1 + e_2 + \dots + e_r$  be a sum of orthogonal primitive idempotents in  $\Lambda V' * G'$  such that  $e\Lambda V' * G'e = \Lambda(G')$  is its basic algebra. Let  $\tilde{e} = \sum_{j=1}^m \sum_{i=1}^r e_i \delta_j = \sum_{j=1}^m e \delta_j$ , then  $\tilde{e}$  is an idempotent in  $(\Lambda V' * G')\# \widehat{C_m}^*$ .

We also have the following lemma.

**Lemma 3.2.**

$$\Lambda(G')\# \widehat{C_m}^* = \tilde{e}((\Lambda V' * G')\# \widehat{C_m}^*)\tilde{e}.$$

*Proof.* Clearly,  $\Lambda(G')\# \widehat{C_m}^* \subseteq \tilde{e}((\Lambda V' * G')\# \widehat{C_m}^*)\tilde{e}$ . For any  $\sum_{t=1}^m a_t \delta_t \in (\Lambda V' * G')\# \widehat{C_m}^*$ , with  $a_t \in \Lambda V' * G'$ , write  $[a_t]_j$  for the  $\omega_j$  component of  $a_t$ , we have that  $\tilde{e}(\sum_{t=1}^m a_t \delta_t)\tilde{e} = \sum_{t=1}^m \sum_{j=1}^m \sum_{j'=1}^m e \delta_j \cdot (a_t \delta_t \cdot e \delta_{j'}) = \sum_{t=1}^m e \sum_{j=1}^m [a_t]_{j-t} e \delta_t = \sum_{t=1}^m (e a_t e) \delta_t$ . Thus

$$\Lambda(G')\# \widehat{C_m}^* \supseteq \tilde{e}((\Lambda V' * G')\# \widehat{C_m}^*)\tilde{e},$$

and hence

$$\tilde{e}((\Lambda V' * G')\# \widehat{C_m}^*)\tilde{e} = \Lambda(G')\# \widehat{C_m}^*.$$

□

Now we describe the bound quiver of  $\Lambda(G')\# \widehat{C_m}^*$  using the bound quiver of  $\Lambda(G)$ .

**Theorem 3.3.** *Let  $(Q_G, \theta_G)$  be the bound quiver of  $\Lambda(G)$  with the Nakayama automorphism  $\nu$ , then the bound quiver  $(Q_{\tilde{G}}, \theta_{\tilde{G}})$  of  $\Lambda(\tilde{G})$  is the quiver defined by the following data:*

1. The vertex set  $Q_{\tilde{G},0} = Q_{G,0} \times \mathbb{Z}/m\mathbb{Z}$ .
2. The arrow set  $Q_{\tilde{G},1} = \bigcup_{t \in \mathbb{Z}/m\mathbb{Z}} (\{(\alpha, t) : (i, l) \rightarrow (j, t) \mid \alpha : i \rightarrow j \in Q_{G,1}\} \cup \{(\beta_i, t) : (i, t) \rightarrow (\nu i, t+1) \mid i \in Q_{G,0}\})$ .
3. The relation set

$$\theta_{\tilde{G}} = \bigcup_{t \in \mathbb{Z}/m\mathbb{Z}} (\{(\varrho, t) \mid \varrho \in \theta_G\} \cup \{(\beta_{\nu(i)}, t+1)(\beta_i, t) \mid i \in Q_{G,0}\} \cup \{(\alpha, t)(\beta_{\nu^{-1}i}, t-1) - (\beta_{\nu^{-1}j}, t-1)(\nu^{-1}\tilde{\sigma}\alpha, t-1) \mid \alpha : i \rightarrow j \in Q_{G,1}\}),$$

here  $(\varrho, t)$  is defined as follows: if  $p = \alpha_l \cdots \alpha_1$  is a path in  $Q_G$ , write  $(p, t) = (\alpha_l, t) \cdots (\alpha_1, t)$  and if  $\varrho = \sum_v h_v p_v$  for paths  $p_v$  of  $Q_G$  and  $h_v \in k$ , write  $(\varrho, t) = \sum_v h_v (p_v, t)$  for all  $t \in \mathbb{Z}/m\mathbb{Z}$ .

4. The Nakayama translation of  $\Lambda(\tilde{G})$  is defined by  $\nu(i, t) = (i, t-1)$ , for  $(i, t) \in Q_{\tilde{G},0}$ .

*Proof.* By Lemma 3.2,  $(Q_{\tilde{G}}, \theta_{\tilde{G}})$  is the bound quiver  $\Lambda(G')\# \widehat{C_m}^*$ .

Since  $\Lambda(G')$  is a graded self-injective algebra of Loewy length  $s+2$ , it is an  $s$ -translation algebra and  $(Q_{G'}, \theta(G'))$  is an admissible  $s$ -translation quiver. The theorem follows from Proposition 2.3 and Proposition 5.5 of [7]. □

We have the following corollary.

**Corollary 3.4.** *Let  $(Q_G, \rho_G)$  be the bound McKay quiver of  $G$  with the Nakayama automorphism  $\nu$ , then the bound quiver of  $\tilde{G}$  is  $(Q_{\tilde{G}}, \rho_{\tilde{G}})$ , where  $Q_{\tilde{G}}$  is as in the above theorem and*

$$\rho_{\tilde{G}} = \bigcup_{t \in \mathbb{Z}/m\mathbb{Z}} (\{(\varrho, t) \mid \varrho \in \rho_G\} \cup \{(\alpha, t)(\beta_{\nu^{-1}i}, t-1) - (\beta_{\nu^{-1}j}, t-1)(\nu^{-1}\alpha, t-1) \mid \alpha : i \rightarrow j \in Q_{G,1}, l \in \mathbb{Z}_m\}).$$

*Proof.* By Theorem 3.3, we get the bound quiver  $(Q_{\tilde{G}}, \theta_{\tilde{G}})$  of  $\Lambda(\tilde{G})$ . Note that  $V(\tilde{G})$  and  $\Lambda(\tilde{G})$  are quadratic dual, similar to the proof of Corollary 2.4, one sees that  $\rho_{\tilde{G}}$  spans the orthogonal subspace of  $\theta_{\tilde{G}}$  in  $kQ_{\tilde{G},2}$ . Thus  $(Q_{\tilde{G}}, \rho_{\tilde{G}})$  is the bound McKay quiver of  $\tilde{G}$ .  $\square$

#### 4. MAIN THEOREMS

In this section we shall prove our main theorem.

Let  $(Q, \rho)$  be a bound quiver and let  $Q'$  be a full bound subquiver of  $Q$  in the sense that the arrows from  $i$  to  $j$  in  $Q'$  are the same as in  $Q$  provided that both  $i, j$  are vertices in  $Q'$ . For each element  $a = \sum_{v \in J_a} h_v p_v \in kQ$ , where  $p_v$  are paths in  $Q$  and  $h_v \in k$ . Let  $J'_a = \{v \in J_a \mid \text{the vertices of } p_v \text{ are all in } Q'\}$  and write  $a_{Q'} = \sum_{v \in J'_a} h_v p_v$ . We say that  $a_{Q'}$  is the component of  $a$  in  $Q'$ . For each  $v \in J_a \setminus J'_a$ , we have that  $p_v$  passes through a vertex outside  $Q'$ , thus we have  $p_v = q_v e_{i_v} q'_v$  for  $i_v$  in  $Q_0 \setminus Q'_0$  in  $kQ$ . Let  $\rho' = \{a_{Q'} \mid a \in \rho\}$  be the relation set on  $Q'$  induced by  $\rho$  and call  $(Q', \rho')$  a *full bound subquiver* of  $(Q, \rho)$ .

Let  $Q'$  and  $Q''$  be finite quivers, write  $s, t$  for the maps from the arrow set to the vertex set sending an arrow to its source and its target, respectively. So we have  $\alpha : s(\alpha) \rightarrow t(\alpha)$  for an arrow  $\alpha$ . A pair of maps  $\omega = (\omega_0, \omega_1)$ , where  $\omega_0 : Q'_0 \rightarrow Q''_0, \omega_1 : Q'_1 \rightarrow Q''_1$ , is called a *quiver embedding* if it satisfies the following conditions:

- (i).  $s\omega_1(\alpha) = \omega_0 s(\alpha), t\omega_1(\alpha) = \omega_0 t(\alpha)$  for  $\alpha \in Q'_1$ .
- (ii).  $\omega_0$  and  $\omega_1$  are injections.

In this case, we write  $\omega(Q') \subseteq Q''$ .

Obviously,  $\omega$  induces an algebra monomorphism  $\omega : kQ' \rightarrow kQ$ .

Let  $(Q', \rho'), (Q, \rho)$  be bound quivers, and  $\omega$  an algebra monomorphism  $\omega : kQ' \rightarrow kQ$ , write  $\omega(\rho') = \{\omega(x) \mid x \in \rho'\}$ . We introduce the following definition.

**Definition 4.1.** *The bound quiver  $(Q', \rho')$  is said to be a truncation from the bound quiver  $(Q, \rho)$ , if there is a quiver embedding  $\omega : Q' \rightarrow Q$  satisfying that  $\omega(\rho')$  is induced by  $\rho$ .*

In this case, the quiver embedding  $\omega$  is called a bound quiver truncation of  $(Q', \rho')$  from  $(Q, \rho)$ , and the algebra  $\Lambda' = kQ'/(\rho')$  is called a truncation of the algebra  $\Lambda = kQ/(\rho)$ .

We have the following proposition.

**Proposition 4.2.** *Let  $\Lambda'$  and  $\Lambda$  be algebras over  $k$  given by the bound quivers  $(Q', \rho')$  and  $(Q, \rho)$ , respectively. If  $\Lambda'$  is a truncation of  $\Lambda$ , there is a subset  $E$  of the orthogonal primitive idempotents of  $\Lambda$  such that  $\Lambda' \cong \Lambda/(E)$ , where  $(E)$  is the ideal generated by  $E$ .*

*Proof.* Since  $\Lambda'$  is a truncation of  $\Lambda$ ,  $(Q', \rho')$  is a truncation from  $(Q, \rho)$ , induced by some quiver embedding  $\omega$  and  $\Lambda' = kQ'/(\rho')$ . Let  $\tilde{E}$  be the preimage of  $E$  in  $kQ$ , that is, the set of trivial paths in  $kQ$  which induce primitive idempotents in  $\Lambda$ . Then we have an isomorphism  $kQ/(\tilde{E}) \simeq kQ'$ , write  $\phi : kQ \rightarrow kQ'$  for the homomorphism which induces the isomorphism, then  $\text{Ker } \phi = (\tilde{E})$ . Let  $\pi' : kQ' \rightarrow \Lambda'$  be the canonical homomorphism, then  $\pi' \phi : kQ \rightarrow \Lambda'$  is an epimorphism and we have that  $\text{Ker } \pi' \phi$  is generated by  $E \cup \omega(\rho')$ , that is.  $\text{Ker } \pi' \phi = (E, \omega(\rho')) = (E, \{a_{\omega(Q')} \mid a \in \rho'\}) = (E, \rho)$ , since  $q e_i q'$  in  $(E)$  if  $i \in E$ . In particular,  $(\rho) \subset \text{Ker } \pi \phi$  and the map

$\pi'\phi = \psi\pi$  factor through the canonical homomorphism  $\pi : kQ \rightarrow kQ/(\rho) = \Lambda$ . Thus  $\psi : \Lambda \rightarrow \Lambda'$  is an epimorphism and  $\text{Ker } \psi = \pi(\tilde{E}) = (E)$ . That is  $\Lambda' \cong \Lambda/(E)$ .  $\square$

Let  $\Gamma^{(n)}$  be an  $n$ -complete algebra over  $k$  which is the cone of an  $n-1$ -complete algebra and let  $(Q_n, \rho_n)$  be its bound quiver with  $n$ -Auslander-Reiten translation  $\tau_n$ . Let  $G \subset GL(n, k)$  be a finite subgroup and let  $(Q_G, \rho_G)$  the McKay quiver of  $G$  with the Nakayama translation  $\nu_G$ . We say that the bound quiver of  $\Gamma^{(n)}$  is a *truncation of the bound McKay quiver of  $G$*  if there is a truncation  $\omega^{(n)}$  of  $(Q_n, \rho_n)$  from  $(Q_G, \rho_G)$  such that

$$\omega_0^{(n)} \tau_n = \nu_G \omega_0^{(n)}.$$

Now we state and prove our main theorem.

**Theorem 4.3.** *Let  $\Gamma^{(n)}$  be an  $n$ -complete algebra over  $k$  which is the cone of an  $n-1$ -complete algebra and let  $\Gamma^{(n+1)}$  be the cone of  $\Gamma^{(n)}$ . Assume that there is a finite subgroup  $G \subset GL(n, k)$  such that the bound quiver of  $\Gamma^{(n)}$  is a truncation of the McKay quiver of  $G$ .*

*Then there is  $m$ , and a finite subgroup  $\tilde{G}$  of  $GL(n+1, k)$  such that the bound quiver of  $\Gamma^{(n+1)}$  is a truncation of the McKay quiver of  $\tilde{G}$ .*

*Proof.* Let  $(Q_n, \rho_n)$  and  $(Q_{n+1}, \rho_{n+1})$  be the bound quiver of  $\Gamma^{(n)}$  and  $\Gamma^{(n+1)}$ , respectively. Then  $(Q_{n+1}, \rho_{n+1})$  is constructed from  $(Q_n, \rho_n)$  by Iyama as in Theorem 1.1.

Let  $m$  be a sufficient large integer such that  $\tau_n^m i = 0$  for all vertex  $x \in Q_{n,0}$ . Let  $G'$  be the subgroup of  $SL(n+1, k)$  which is the image of  $G$  under the canonical embedding  $g \rightarrow \begin{pmatrix} g & \\ & \det^{-1}(g) \end{pmatrix}$ , and let  $C_m$  be the subgroup of  $GL(n+1, k)$  generated by  $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \zeta_m \end{pmatrix}$ , where  $\zeta_m$  is a primitive  $m$ th root of the unity.

Let  $\tilde{G} = G' \times C_m$  be a finite subgroup of  $GL(n+1, k)$ . The bound McKay quiver of  $\tilde{G}$  is given in Corollary 3.4.

Let  $\omega$  be a truncation of the bound quiver of  $\Gamma^{(n)}$  from the McKay quiver of  $G$ . Now we define a quiver embedding  $\tilde{\omega} = (\tilde{\omega}_0, \tilde{\omega}_1)$  from  $(Q_{n+1}, \rho_{n+1})$  to the bound McKay quiver  $(Q_{\tilde{G}}, \rho_{\tilde{G}})$ : Define  $\tilde{\omega}_0$  on the vertex set by

$$\tilde{\omega}_0(x, d) = (\omega_0(x), -d)$$

for  $(x, d) \in Q_{n+1,0}$ . For an arrow  $\alpha : x \rightarrow y$  in  $Q_{n,1}$  and  $d \geq 0$  with  $\tau_n^d x \neq 0 \neq \tau_n^d y$ ,  $(\alpha, d)$  is an arrow of the first type in  $Q_{n+1,1}$ , we define  $\tilde{\omega}_1(\alpha, d) = (\omega_1(\alpha), -d)$ . For each vertex  $x$  and  $d \geq 0$  with  $\tau_n^{d+1} x \neq 0$ ,  $(x, d)_1$  is an arrow of the second type in  $Q_{n+1,1}$ , we define  $\tilde{\omega}(x, d)_1 = (\beta_{\omega_0(x)}, -d)$ . One sees easily that  $\tilde{\omega}$  is a quiver embedding from  $Q_{n+1}$  to  $Q_{\tilde{G}}$ .

For a relation of the form  $(r, d) \in \rho_{n+1}$  with  $r \in \rho_n$ ,  $\omega(r)$  is the component in  $\omega(Q_n)$  of some relation  $\varrho \in \rho_G$ . Thus  $\tilde{\omega}(r, d) = (\omega(r), -d)$  is the component in  $\omega(Q_{n+1})$  of the relation  $(\varrho, d) \in \rho_{\tilde{G}}$ . For a relation of the form  $(\alpha, d-1)(x, d)_1 - (\tau_n^- y, d)_1(\tau_n^- \alpha, d)$  with  $\alpha : \tau_n x \rightarrow y \in Q_{n,1}$  and  $d > 0$ , write  $\omega(\alpha) = \alpha$ ,  $-d = t-1$

then

$$\begin{aligned} & \tilde{\omega}((\alpha, d-1)(x, d)_1 - (\tau_n^- y, d)_1(\tau_n^- \alpha, d)) \\ &= (\omega(\alpha), -d+1)\tilde{\omega}(x, -d)_1 - \tilde{\omega}(\tau_n^- y, -d)_1(\omega\tau_n^-(\alpha), -d) \\ &= (\alpha, t)(\beta_x, t-1) - (\beta_{\nu^{-1}y}, t-1)(\nu^{-1}\alpha, t-1), \end{aligned}$$

which is in  $\rho_{\tilde{G}}$ . By comparing Theorem 1.1 with Theorem 3.3 and Corollary 3.4, we see that  $\tilde{\omega}$  is a truncation of the bound quiver of  $\Gamma^{n+1}$  from the McKay quiver of  $\tilde{G}$ .  $\square$

Fix an integer  $s \geq 1$ , Iyama describes a family of  $(n-1)$ -Auslander absolute  $n$ -complete algebras  $T_s^{(n)}(k)$  using quivers with relations, for  $n = 1, \dots, s$  [12]. Let  $A_s$  be a quiver with  $A_{s,0} = \{1, \dots, s\}$ ,  $A_{s,1} = \{a_i : i \rightarrow i+1 \mid i = 1, \dots, s-1\}$ , then  $A_s$  is a linear oriented Dynkin quiver of type  $A$ . Suppose that  $T_s^n(k) \cong (kQ^n)/(\rho_n)$ . As an immediate application of Theorem 4.3, we give a new proof of the first part of Theorem 4.6 of [5].

**Example 4.4.** Let  $T_s^n(k)$  be an  $(n-1)$ -Auslander absolute  $n$ -complete algebra for  $n \geq 1$ , then there exists a finite abelian group  $G_n \subset GL(n, k)$ , such that the bound quiver of  $T_s^n(k)$  is a truncation of the bound McKay quiver  $(Q_{G_n}, \rho_{G_n})$  of  $G_n$ .

*Proof.* For  $n = 1$ ,  $T_s^1(k) \cong kA_s$ . Let  $G_1 = \mathbb{Z}_{r_1} \subseteq GL(1, k)$ , where  $r_1$  is sufficiently large. By Proposition 3.2 of [5],  $Q_{G_1,0} = \mathbb{Z}_{r_1}$  and  $Q_{G_1,1} = \{\alpha_i : i \rightarrow i+1 \mid i \in \mathbb{Z}_{r_1}\}$ . We define the map:  $\varsigma_1 : A_s \rightarrow Q_{G_1}$  as follows:  $\varsigma_1(i) = i$ ,  $\varsigma_1(a_i) = \alpha_i$ , for  $i \in A_{s,0}$ ,  $a_i \in A_{s,1}$ . It's straightforward to check that  $\varsigma_1$  is a quiver homomorphism, and observe that  $\rho_{A_s} = \rho_{G_1} = \emptyset$ . Then  $A_s$  is a truncation of  $(Q_{G_1}, \rho_{G_1})$  induced by  $\varsigma_1$ , and  $\nu_1 \varsigma_1(i) = i-1 = \varsigma_1 \tau_1(i)$ ,  $\nu_1 \varsigma_1(a_i) = \alpha_{i-1} = \varsigma_1 \tau_1(a_i)$ , for  $i \in A_{s,0}$ ,  $a_i \in A_{s,1}$ , where  $\nu_1$  is the Nakayama translation defined on the McKay quiver  $Q_{G_1}$ .

By Theorem 1.19 of [12],  $T_s^n(k)$  is the cone of  $T_s^{n-1}(k)$ , and  $T_s^{n-1}(k)$  is an  $(n-1)$ -complete algebra. Then the claim follows by induction on  $n$  and Theorem 4.3.  $\square$

## REFERENCES

- [1] Assem, I., Simson, D., Skowronski, A., Element of the representation theory of associative algebras. Volume 1 techniques of representation theory, Cambridge University Press, Cambridge, 2006.
- [2] Anderson F. W., Fuller, K. R., Rings and categories of modules, Graduate texts in Mathematics 13, Springer-Verlag, New York, Heidelberg, Berlin, 1973 (new edition 1991).
- [3] Fossum, R.M., Griffith, P.A., Reiten, I. Trivial extensions of Abelian categories, Lect. Notes in Math. 456(1975), Springer-Verlag, Berlin-Heidelberg-New York.
- [4] Guo, J. Y., Coverings and Truncations of Graded Self-injective Algebras. J. Algebra, 2012, 355(1): 9-34.
- [5] Guo, J. Y., McKay quivers and absolute  $n$ -complete algebras, Science China Mathematics, 2013, 56: 1607-1618.
- [6] Guo, J. Y., Matínez-Villa, R., Algebras pairs associated to McKay quivers. Comm. Algebra, 2002, 30(2): 1017-1032.
- [7] Guo, J. Y., On  $n$ -translation algebras, preprint, 2014, arxiv: arxiv.org/abs/1406.6136.
- [8] Guo, J. Y., Yin, Y., Zhu, C., Returning arrows for self-injective algebras and Artin-Schelter regular algebras. J. Alg, 2014, 397: 365C378.
- [9] Guo, J. Y., On McKay quiver and covering space (in Chinese). SciSci China Ser, 2011, 41(5): 393-402.
- [10] Iyama, O., Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories. Adv. Math, 2007, 210: 22-50.
- [11] Iwanaga, Y., Wakamatsu, T. Trivial extension of Artin algebra, LNM 832. Berlin-Heidelberg-New York: Springer-Verlag, 1980: 3295-301.
- [12] Iyama, O., Cluster tilting for higher Auslander algebras, Adv. Math, 2011, 226: 1-61.

- [13] Montgomery, S., Passman, D. S., Algebraic analogs of the Connes spectrum. J. Alg, 1988, 115(1): 92-124.
- [14] Passman, D. S., Group rings, crossed products and Galois theory. CBMS Regional Conference Series in Mathematics 64 AMS Providence, 1986.
- [15] Skowronski, A. Yamagata, K. Frobenius Algebras I: basic representation theory, EMS Textbk. Math., European Mathematical Society, Zurich, 2011.
- [16] Zheng, L. J., Twisted trivial extension and representation dimension. Advances in Mathematics(China) (in Chinese), 2014, 4: 512-520.

COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, KEY LABORATORY OF HPSCIP (MINISTRY OF EDUCATION OF CHINA),, HUNAN NORMAL UNIVERSITY, CHANGSHA 410081, CHINA

COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, KEY LABORATORY OF HPSCIP (MINISTRY OF EDUCATION OF CHINA),, HUNAN NORMAL UNIVERSITY, CHANGSHA 410081, CHINA

SCHOOL OF MATHEMATICS AND PHYSICS, UNIVERSITY OF SOUTH CHINA, HENGYANG, 421001, HUNAN, P. R. CHINA